

Geometry and topological order in the non-relativistic Luttinger liquid.

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We demonstrate that the notion of spin-charge separation as follows from the Tomonaga-Luttinger theory of relativistic fermions does not suffice to completely specify the Luttinger liquid state associated with the fermions of condensed matter physics. The latter carries in addition a form of topological order which can be made visible using non-local correlation functions familiar from lattice gauge theory.

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Systems composed of an infinite number of interacting quantum particles can be characterized by long wavelength excitations carrying quantum numbers unrelated to those of the constituents. A classic example is the phenomenon of spin-charge separation which is well established in one dimensional electron systems [1]. The elementary excitations can be viewed as pieces of the original electron, carrying separately the charge and the spin. This notion first emerged when it was demonstrated that systems of relativistic fermions in 1+1D can be represented exactly under a variety of circumstances in terms of Bose fields, carrying separately the spin and the charge degrees of freedom (bosonization). Although a-priori it is unclear whether the relativistic theory has anything to say about systems of non-relativistic electrons, an overwhelming case emerged that bosonization is correctly describing the asymptotic properties of the local correlation functions of the latter. However, the loophole is that these might differ by a non-critical topological order which cannot be detected using two-point correlators in terms of local operators. Here we will demonstrate that at least the Hubbard model in 1+1D carries such an order which is alien to relativistic fermions. We believe that the Hubbard model is in this regard fully representative for all non-relativistic electron systems.

This hidden order has to do with the fact that the charge carriers ('holons') are at the same time Ising 'domain walls' (kinks) in the spin system, a property shared with stripes in two dimensional systems [2]. Resting on the *geometrical* definition of spin-charge separation as deduced from the Bethe-Ansatz [3] solution of the Hubbard model at infinite U , obtained first by Woyanovich [4] and later Ogata and Shiba [5], we construct a non-local operator measuring directly the topological order. Using a state of the art DMRG method, we demonstrate numerically that this order is present in the Hubbard chain for all positive U 's and electron densities. We subsequently show that a straightforward application of bosonization leads to inconsistencies, demonstrating that this topological order is not present in the Dirac vacuum in 1+1D. Our non-local correlator has the structure of the Wil-

son loop of a Z_2 gauge theory and we conclude with the speculation that a specific kind of 1+1D superconductor might be governed by a local Ising symmetry.

Some time ago [4,5], a simple but most peculiar property of the Bethe-Ansatz solution of the 1D Hubbard model in the limit that $U \rightarrow \infty$ was found; namely that the wave function factorizes into a charge part ψ_{SF} , depending only on where the electrons are, and a spin part ψ_H which merely depends on the way the spins are distributed,

$$\psi(\{x_i\}_{i=1}^N; \{y_j\}_{j=1}^{N/2}) = \psi_{SF}(\{x_i\}_{i=1}^N) \psi_H(\{y_j\}_{j=1}^{N/2}). \quad (1)$$

The charge part ψ_{SF} is the wave-function of a non-interacting spinless-fermion system where the coordinates x_i refer to the positions of the electrons/spinless fermions. The spin part ψ_H is identical to the wave-function of a chain of Heisenberg spins interacting via a nearest-neighbor antiferromagnetic exchange. In ψ_H only the positions of the up spins are needed and these correspond with the coordinates y_i . However, the miracle is that the coordinates y_i do not refer to the Hubbard chain, but instead to a *new space*: a lattice with sites at coordinates x_1, x_2, \dots, x_N given by the positions of the charges in a configuration with amplitude ψ_{SF} .

This shows that the quantum dynamics of interacting electrons generates a *geometrical structure* analogous to the fabric of general relativity. Let us visualize this for a representative example (Fig. 1). Consider N electrons on a chain with L sites under the condition that $N < L$ such that the charge configurations can be specified by the locations of the holes. A charge configuration in the full Hubbard chain ('external space'), has an amplitude ψ_{SF} in the wavefunction with the coordinates of the dots corresponding with the x_i 's. The spin system sees a different 'internal space' obtained from the full space by removing the holes *together with the sites where the holes are located*, substituting the hole and its site with an antiferromagnetic exchange between the sites neighboring the hole (the 'squeezed space' [5]).

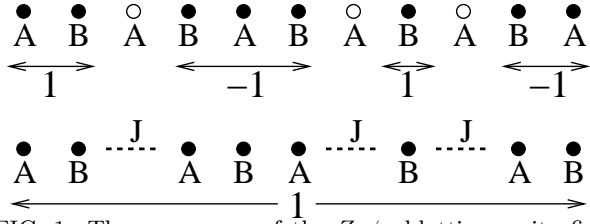


FIG. 1. The emergence of the Z_2 'sublattice parity field' in the geometrical 'squeezing' operation. The open circles refer to some configuration of holes. The black dots define the embedding space of the spin system in both the full- (upper) and squeezed (lower) lattice. The unsqueezing operation can be parametrized in terms of binding of the electric charge quantum to flips in the Z_2 valued sublattice parity field.

This is a very simple geometrical structure which can be formulated in terms of a simple topological 'gauge' theory. In what regards are the full chain and the squeezed chain different? The squeezed chain is obviously shorter than the full chain and this is a simple dilation: a distance x measured in the full chain becomes a distance ρx in the squeezed chain ($\rho = N/L$, the electron density) when $x \gg 1$, the lattice constant. The other aspect is also simple, but less trivial in its consequences. The spin system is a quantum-antiferromagnet and is therefore sensitive to the geometrical property of bipartiteness. A lattice is called bipartite if it can be subdivided in two sublattices A and B , such that all sites on the A sublattice are neighbored by B sublattice sites and vice versa. This division can be done in two ways ($\cdots - A - B - A - B \cdots$ and $\cdots - B - A - B - A \cdots$) defining a Z_2 valued quantity 'sublattice parity', $p = \pm 1$. Consider now what happens with sublattice parity in the squeezing. For the Heisenberg spin chain a redefinition of $p = 1 \leftrightarrow -1$ does not carry any consequence ('pure gauge'). However, sublattice parity becomes important in the mapping of squeezed space into full space (Fig. 1). 'Fix the gauge' in squeezed space by choosing a particular sublattice parity, and consider what happens when it is unsqueezed. The holes are inserted, and because every hole is attached to one site, every time a hole is passed the sublattice parity flips. This is true for every instantaneous charge configuration, but the ground state is a superposition of many of these configurations: charge is fluctuating and since it is attached to the sublattice parity flips, the full space which is observable by external observers (experimentalists) should be considered as a fluctuating geometry. However, this is a very simple fluctuating geometry because all that is fluctuating is the property of bipartiteness.

Let us now ask the question if a correlation function can be defined acting on the full Hubbard chain which can measure the true spin correlations living in squeezed space. Since all that matters is sublattice parity this can be achieved by simply multiplying the spin operator by a factor -1 every time a hole is passed, thereby remov-

ing the sublattice parity flips from the spin correlations. Define staggered magnetization as $\vec{M}(x_i) = (-1)^{x_i} \vec{S}(x_i)$ where \vec{S} is the spin operator ($S^z = \frac{1}{2}(n_{\uparrow} - n_{\downarrow})$, $S^+ = c_{\uparrow}^\dagger c_{\downarrow}$) with the charge operator $n_{x_i} = n_{x_i\uparrow} + n_{x_i\downarrow}$ taking the values 0, 1 and 2 for an empty-, singly- and doubly occupied sites respectively. The correlation function we are looking for is [6],

$$O_{top}(x) = \langle M^z(0) (-1)^{\sum_{j=1}^{x-1} 1-n_j} M^z(x) \rangle. \quad (2)$$

The operator $(-1)^{(1-n_j)}$ takes the value $+1$ for a singly occupied site while it is -1 for a charge (hole, or doubly occupied) site. By multiplying these values on the interval $0 < j < x$ all the minus signs associated with the sublattice parity flips are removed from the spin correlations.

Although the 'string' operator $(-1)^{\sum_{j=1}^{x-1} 1-n_j}$ is non-local it can be evaluated in a straightforward manner using the techniques introduced by Parola and Sorella [7]. It is easily shown that in the large U limit,

$$\begin{aligned} \langle M^z(0) (-1)^{\sum_{j=1}^{x-1} 1-n_j} M^z(x) \rangle \\ = \sum_{j=2}^{x+1} P_{SF}^x(j) (-1)^j O_{Heis.}(j-1) \end{aligned} \quad (3)$$

where $O_{Heis.}$ is the spin correlator of the Heisenberg chain, while $P_{SF}^x(j) = \langle n(0) n(x) \delta(\sum_{l=0}^x n_l - j) \rangle_{SF}$ is the probability of finding j spinless fermions in the interval $[0, x]$. This factor causes the additional decay of the spin correlations due to the charge fluctuations in the standard spin correlator. However, it is easily shown that it is precisely compensated in O_{top} by the factor $(-1)^j$ coming from the string operator and we find the result, asymptotically exact for large distances x ,

$$O_{top}(x) = \frac{\rho}{x} \ln^{1/2}(\rho x). \quad (4)$$

Which is identical to the result for the pure Heisenberg spin chain $\frac{1}{x} \ln^{1/2}(x)$ after rescaling the amplitude of staggered spin $\vec{M} \rightarrow \frac{\vec{M}}{\rho}$ and the measure of length $x \rightarrow x/\rho$, where ρ is the average charge density. Comparing Eq.(4) with the well-known [7] asymptotic behavior of the local staggered spin correlations for the Hubbard model, $O_{\vec{M}}(x) = \langle |M^z(0) M^z(x)| \rangle \sim \frac{\cos([2k_F - \pi]x)}{x^{1+K_\rho}}$ we find that the former decays more slowly. The charge-stiffness K_ρ is associated with the decay of the charge correlations, $\langle n(0) n(x) \rangle \sim \cos([4k_F - \pi]x)/x^{4K_\rho}$, and $K_\rho > 0$ for all $\rho \neq 1$. Hence, the charge fluctuations modify the spin correlations because the charge is attached to the sublattice parity flips, resulting in a simple multiplicative factor $1/x^{K_\rho}$.

Is the above an accident of the strongly coupled case? Having identified the operator Eq. (2) which in combination with the local spin-spin correlator measures directly the presence or absence of the topological order

it becomes possible to study it in any one dimensional system. For finite U 's it becomes very hard to calculate Eq. (2) from the Bethe-Ansatz solution and therefore we compute it numerically for the Hubbard model using the density matrix renormalization group. We employ an algorithm recently developed by one of us [8], making explicit use of the full non-abelian $SO(4)$ symmetry of the Hubbard model. In this formulation, the generators of the global symmetry group are the spin \vec{S} and the pseudospin \vec{I} ; the latter is a generalization from the charge $U(1)$ to an $SU(2)$ symmetry such that the particle number at site i is given by $n_i = 1 + 2I_i^z$.

In principle, it is straightforward to calculate the expectation value of string operators such as Eq. (2) using the DMRG method. However, the calculation of critical correlators with DMRG is not unproblematic even in 1+1D, because eventually the truncation errors inherent in the form of the wavefunction will cause an exponential decay [9], and the requirement to use open boundaries causes a loss of translational invariance even at large distances. We used a relatively large system size (1000 sites) to reduce the effect of the boundary conditions, and used a basis size (700 $SO(4)$ states) large enough to achieve a truncation length of the order of 200-300 lattice constants. In this way meaningful results can be obtained from a simple curve fit to obtain the desired exponents.

In Fig. 2 we show our results for the exponents of both the standard two point spin correlator $\langle \vec{M}(0)\vec{M}(x) \rangle$, and our topological correlator, Eq. (2), as a function of density for various interaction strengths. The former illustrates the quality of the calculation; the dominant correlation is at the wavevector $2k_F - \pi$, $\sim \cos([2k_F - \pi]x)/x^\eta$ and it is seen from Fig. 2 that the exponent η depends strongly on the parameters. In fact, the exponent behaves exactly according to the expectations: $\eta = K_\sigma + K_\rho$ where $K_\sigma = 1$ and K_ρ is consistent with the values previously obtained by Schulz [10] for the same values of U and ρ . Considering now the topological correlator, the dominant component lives at the wavevector $q = 0$; we find no other characteristic momenta in the Fourier transform of Eq. (2) which is already reminiscent of the staggered spin correlator of a Heisenberg chain. This is further amplified by our finding that the exponent η_{top} does not depend on the microscopic parameters at all. In fact, for all parameters the exponent of the topological correlator $\eta_{top} = 1 = K_\sigma$. Hence, regardless the values of U and ρ the long distance behavior of the topological correlator is indistinguishable from the spin-spin correlator of a Heisenberg chain, demonstrating that the Hubbard model indeed carries the sublattice parity topological order fully for all values $U > 0$.

Let us now explain the reasons why bosonization fails in principle to describe the topological order. Tomonaga-Luttinger bosons cannot carry topological order for the elementary reason that these parametrize a relativistic theory. At the center of our considerations for the lat-

tice model is the fact that the topological structure is Z_2 valued (the sublattice parity) and this in turns rests on the fact that on a lattice the number operator n_i is integer valued: a site is either empty, singly or doubly occupied corresponding with charge quantum numbers 0,1, and 2; by exponentiation one takes out the evenness or unevenness by the factors $\sigma_i^3 = (-1)^{n_i} = \pm 1$. In a relativistic theory one has to keep the fermion creation- and annihilation operator apart, $n_r \rightarrow \psi^\dagger(r)\psi(r+\epsilon)$ and the resulting Schwinger terms are the building blocks for bosonization. Thereby, the charge density becomes the gradient of a continuous function $n(x) = (1/\pi)\partial_x \phi$. Accordingly, $(-1)^{n_i} \sim (-1)^{\partial_x \phi}$ is a complex scalar and not an Ising valued operator. This reflects the fact it is impossible to localize an electron in space in a relativistic theory, while the ability to ascribe a position to the electron is at the heart of the squeezing procedure on the lattice.

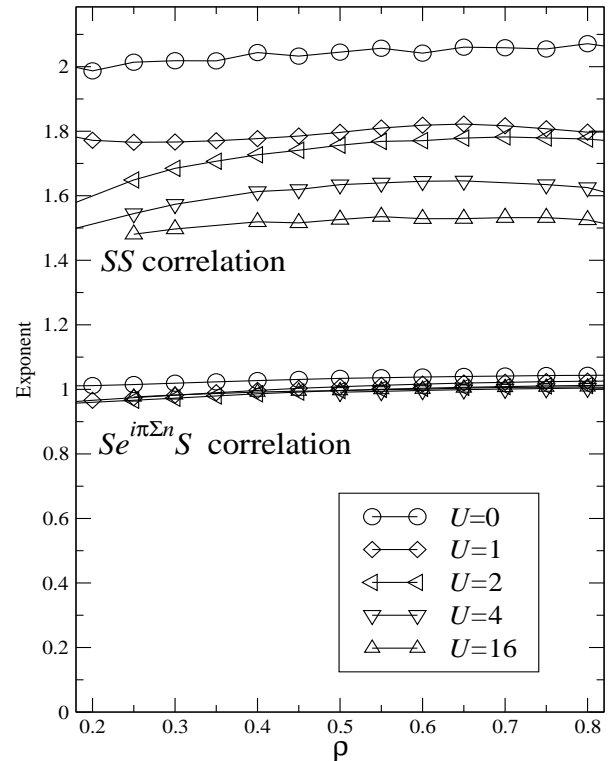


FIG. 2. Exponents of the spin-spin correlation, η , (top) and the topological correlation, η_{top} , (bottom) as a function of electron filling, for various values of interaction U .

In fact, this problem is to a certain extent repairable. The operator entering the string correlator is not σ^3 itself, but instead $\Pi_l \sigma^3 = \exp(i\pi \int_x^{x'} dx n(x))$, taking the continuum limit. The integral smooths out the discreteness, and an integral over the continuous bosonization fields can be manipulated to mimic the product of Ising

valued operators. This is a subtle UV regularization issue, which we will discuss elsewhere in detail. To give the reader some feeling, let us consider the expectation value of the string correlator itself, $D(x) = \langle (-1)^{\sum_{j=0}^x n_j} \rangle = \langle \cos \left[\pi \sum_j n(j) \right] \rangle \rightarrow \langle \cos \left[\pi \int_0^x dy n(y) \right] \rangle$. According to bosonization, the total charge density is $n(x) = \sqrt{\frac{2}{\pi}} \frac{\partial \varphi_c}{\partial x}(j) + \mathcal{O}_{CDW}(j) + \mathcal{O}_{CDW}^\dagger(j)$ where $\mathcal{O}_{CDW}(x) = \exp(-2ik_F x) \exp(i\sqrt{2\pi}\varphi_c) \cos(\sqrt{2\pi}\varphi_s)$. It follows that $D(x) \sim \langle \exp(i\sqrt{2\pi}[\varphi_c(x) - \varphi_c(0)]) \rangle = \cos(2k_F x)/x^{K_\rho}$. On the other hand, because on the lattice $(-1)^{n_i} \equiv (-1)^{2S_i^z}$ we might as well calculate $D(x) = \langle \cos \left[2\pi \sum_{j=0}^x S^z(j) \right] \rangle$ and using that in bosonization $2S^z(x) = \sqrt{\frac{2}{\pi}} \frac{\partial \varphi_s}{\partial x}(j) + \mathcal{O}_{SDW}(j) + \mathcal{O}_{SDW}^\dagger(j)$ it follows that $D(x) = \frac{1}{x^{K_\sigma}}$. For $U = 0$, $K_\sigma = K_\rho$ and the two expressions are the same, but away from this point $K_\sigma \neq K_\rho$ and bosonization runs into a paradox: depending the way one calculates D one obtains different, and mutually exclusive answers. In fact, the DMRG result shows that $D(x) = A/x^{K_\sigma} + B \cos(2k_F x)/x^{K_\rho}$ (A, B are non-universal amplitudes), and apparently the two ways of calculating $D(x)$ in bosonization recover part of the correct answer. A similar problem occurs when one uses bosonization to calculate the expectation value of $\Pi_l(-1)^{2n_l}$ which is on the lattice just the identity operator; according to bosonization this decays like $\frac{1}{x^{4K_\rho}}$! As we will discuss elsewhere [11], these problems can be traced back to spurious contributions associated with the ‘spreading’ of the electron number density.

In summary, we have discovered a correlation function which makes it possible to measure directly the presence of a hidden order underlying spin-charge separation in 1+1D electron systems. This hidden order can be viewed as either a form of topological order, or as a geometrical structure where the spin system lives in a squeezed space different from the full chain. The suspicion was widespread that the Ogata-Shiba construction was special to the $U \rightarrow \infty$ limit and our main result is the numerical demonstration that squeezed space is generic in the scaling limit. We perceive this as a deep insight, powerful enough to hint at the existence of hitherto unidentified states of 1+1D matter. An exciting possibility is closely related to recent ideas regarding a potential connection between Ising gauge theory and the destruction of stripe order in 2+1 dimensions [2,12]. As we discussed, $(-1)^{n_i}$ is Ising valued and the string operator can be written accordingly as $\Pi_i \sigma_i^3$: this is just the Wilson line of Z_2 gauge theory [13]. Hence, if local Ising symmetry would be dictating, $\langle S \Pi \sigma^3 S \rangle \sim \exp(x/\xi_g) \langle SS \rangle$ because the l.h.s. is gauge invariant while the r.h.s. is not; ξ_g is the length scale where the gauge invariance emerges. In the Luttinger liquid the two are related by an algebraic factor $1/x^{K_\rho}$ but this is due to the algebraic order in the charge system. Therefore, under the conditions that (a) the spin

system stays separated from the charge, (b) the charge quanta are bound to sublattice parity flips, and (c) the charge correlations become short range, one will obtain the required relations between the correlation functions signaling the Z_2 gauge invariance. Condition (c) is generally satisfied when the superconducting phase is forced to condense by applying an external Josephson field. The attractive U Hubbard model fails in this regard because its ground state can be viewed as a continuation of the local pair limit where the spin system is destroyed. However, it appears [14] that the conditions might well be satisfied in $t - J$ ladders under the influence of an external Josephson field, and these have in turn much more similarity with the situation in 2+1D.

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